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Fully Entangled Quantum States in $\mathbf{C}^{N^2}$ and Bell Measurement

Yorick Hardy,[1] Willi-Hans Steeb,[1,3] and Ruedi Stoop[2]

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Entangled quantum states are an important component of quantum computing techniques such as quantum error-correction, dense coding, and quantum teleportation. We describe how to generate fully entangled states in the Hilbert space $\mathbf{C}^N \otimes \mathbf{C}^N$ starting from a unitary matrix and show that they form an orthonormal basis in this space.

KEY WORDS: entanglement; Bell measurement; phase operator.

Entanglement (Hardy and Steeb, 2001; Nielsen and Chuang, 2000; Preskill, 2000; Schrödinger, 1935; Steeb and Hardy, 2000, 2002) is the characteristic trait of quantum mechanics that enforces its entire departure from classical lines of thought. We consider entanglement of pure states. For example in the product Hilbert space $\mathbf{C}^2 \otimes \mathbf{C}^2$ the Bell states

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle), \quad |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle), \quad |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle)$$

are fully entangled states and form an orthonormal basis in $\mathbf{C}^4$. Here $\{|0\rangle, |1\rangle\}$ is an arbitrary orthonormal basis in the Hilbert space $\mathbf{C}^2$. If we choose

$$|0\rangle = \begin{pmatrix} e^{i\phi}\cos\theta \\ \sin\theta \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} -e^{i\phi}\sin\theta \\ \cos\theta \end{pmatrix}$$

[1] International School for Scientific Computing, Rand Afrikaans University, Auckland Park, South Africa.
[2] Institut für Neuroinformatik, ETHZ, Zürich, Switzerland.
[3] To whom correspondence should be addressed at International School for Scientific Computing, Rand Afrikaans University, P.O.Box 524, Auckland Park 2006, South Africa; e-mail: whs@na.rau.ac.za.

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we obtain

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{2i\phi} \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad |\Phi^-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{2i\phi}\cos(2\theta) \\ e^{i\phi}\sin(2\theta) \\ e^{i\phi}\sin(2\theta) \\ -\cos(2\theta) \end{pmatrix}$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -e^{2i\phi}\sin(2\theta) \\ e^{i\phi}\cos(2\theta) \\ e^{i\phi}\cos(2\theta) \\ \sin(2\theta) \end{pmatrix}, \quad |\Psi^-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ e^{i\phi} \\ -e^{i\phi} \\ 0 \end{pmatrix}.$$

If we choose $\phi = 0$ and $\theta = 0$ which simply means we choose the standard basis for $|0\rangle$ and $|1\rangle$ (i.e. $|0\rangle = (1\ 0)^{\mathrm{T}}$ and $|1\rangle = (0\ 1)^{\mathrm{T}}$ we find that the Bell states take the form

$$|\Phi^+\rangle := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad |\Phi^-\rangle := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$|\Psi^+\rangle := \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad |\Psi^-\rangle := \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

We describe how to generate fully entangled states in the Hilbert space $\mathbf{C}^N \otimes \mathbf{C}^N = \mathbf{C}^{N^2}$ and show that they form an orthonormal basis in this space.

Consider the Hilbert space $\mathbf{C}^N$. Let

$$\{|\phi_k\rangle : k = 0, 1, \ldots, N-1\} \tag{1}$$

be an orthonormal basis in C^N. Thus $\langle \phi_j | \phi_k \rangle = \delta_{jk}$ and

$$\sum_{k=0}^{N-1} |\phi_k\rangle\langle\phi_k| = I_N \tag{2}$$

where I_N is the $N \times N$ unit matrix. The last relation is the completeness relation. Next we define the matrix

$$U := \sum_{k=0}^{N-2} |\phi_k\rangle\langle\phi_{k+1}| + |\phi_{N-1}\rangle\langle\phi_0|. \tag{3}$$

Thus we can also write

$$U := \sum_{k=0}^{N-1} |\phi_{k \bmod N}\rangle\langle\phi_{k+1 \bmod N}|. \tag{4}$$

For example, in $\mathbf{C}^2$ we obtain the Pauli matrix σ_x if $|\phi_0\rangle = (1, 0)^{\mathrm{T}}$ and $|\phi_1\rangle = (0, 1)^{\mathrm{T}}$ and the Pauli matrix σ_z if $|\phi_0\rangle = \frac{1}{\sqrt{2}}(1, 1)^{\mathrm{T}}$ and $|\phi_1\rangle = \frac{1}{\sqrt{2}}(1, -1)^{\mathrm{T}}$.

The set of matrices $\{U, U^2, \ldots, U^N\}$ form a commutative group under matrix multiplication where $U^N = I_N$. We also have $\mathrm{tr}(U) = 0$ and $\det(U) = -1$ if N is even and $\det(U) = 1$ if N is odd. The eigenstates of the unitary matrix U satisfy

$$U|\theta_j\rangle = \exp(-i\theta_j)|\theta_j\rangle, \quad j = 0, 1, \ldots, N-1 \tag{5}$$

where $\theta_j := 2\pi j/N$. Thus the set of eigenvalues $\{\exp(-i\theta_j) : j = 0, 1, \ldots, N-1\}$ form a commutative group under multiplication. The group given above and this group are isomorphic. For $N \to \infty$ we have the Lie group U (1).

For the standard basis in $\mathbf{C}^N$ the matix U is given by

$$U = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}. \tag{6}$$

Then the set of matrices $\{U, U^2, \ldots, U^N\}$ is a subgroup of the group of all $N \times N$ permutation matrices under matrix multiplication.

Schwinger (2000) showed that the $N \times N$ matrices U (given by (6)) and the $N \times N$ diagonal matrix

$$Z := \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & \zeta & 0 & \ldots & 0 \\ 0 & 0 & \zeta^2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \zeta^{N-1} \end{pmatrix}.$$

where $\zeta^N = 1$ satisfy the relation $UZ = \zeta ZU$. Furthermore the set of matrices

$$\{U^k Z^j : k, j = 0, 1, \ldots, N-1\}$$

provide a basis in the Hilbert space (Steeb, 1998) of all $N \times N$ matrices over $\mathbf{C}$ with the scalar product $\langle A|B\rangle := \mathrm{tr}(AB^*)/N$. Schwinger (2000) also noticed that if one goes from the discrete finite dimensional case to the continuous case the position-momentum description is recovered.

The normalized eigenvectors of U given by (3) are

$$|\theta_j\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \exp(-i2\pi jk)|\phi_k\rangle \tag{7}$$

where $j = 0, 1, \ldots, N-1$. If we consider U given by (6) (i.e. the standard basis is selected), then we find the eigenvectors

$$\frac{1}{\sqrt{N}}(1, \exp(-i2\pi j/N), \exp(-i4\pi j/N), \ldots, \exp(-i2(N-1)j/N))^{\mathrm{T}}.$$

Thus for the eigenvalue 1 of U we find the normalized eigenvector

$$\frac{1}{\sqrt{N}}(1, 1, \ldots, 1)^{\mathrm{T}}.$$

The eigenstates $\{|\theta_j\rangle : j = 0, 1, \ldots, N-1\}$ and the orthonormal basis given above are connected by the discrete Fourier transform

$$|\theta_j\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \exp(-ik\theta_j)|\phi_k\rangle, \quad |\phi_k\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \exp(ik\theta_j)|\theta_j\rangle. \tag{8}$$

Next we introduce the two matrices

$$\hat{n} := \sum_{k=0}^{N-1} k|\phi_k\rangle\langle\phi_k|, \quad \hat{\theta} := \sum_{j=0}^{N-1} \theta_j|\theta_j\rangle\langle\theta_j|. \tag{9}$$

We have $[U, \hat{\theta}] = 0$. The hermitian matrix $\hat{n}$ can be considered as the number operator with the eigenvalues $0, 1, 2, \ldots, N-1$. The matrix $\hat{n}$ is diagonal in the standard basis. The matrix $\hat{\theta}$ is called the Pegg–Barnett phase operator (Pegg and Barnett, 1997). We note that an outstanding problem in quantum mechanics is the search for a "proper" phase operator. A number of theories for such operators have been proposed, but most of them succumb to one or more of three shortcomings: (i) the operator is non-selfadjoint, (ii) no scheme for an experimental realization, (iii) the operator is operationally defined, leaving the questions open as to what observable the measurement apparatus really represents and what the conjugate observable is (Trifonov *et al.*, 2000).

The matrix $\hat{n}$ has the property

$$\exp(\pm i\theta_j\hat{n})|\phi_k\rangle = \exp(\pm i\theta_j k)|\phi_k\rangle, \quad \exp(\pm i\theta_j\hat{n})|\theta_k\rangle = |\theta_{k\mp j \bmod N}\rangle. \tag{10}$$

The matrix $\hat{\theta}$ has the properties

$$\exp(\pm ik\hat{\theta})|\theta_j\rangle = \exp(\pm ik\theta_j)|\theta_j\rangle, \quad \exp(\pm ik\hat{\theta})|\phi_m\rangle = |\phi_{m\mp k \bmod N}\rangle. \tag{11}$$

Note that the matrices $\hat{n}$ and $\hat{\theta}$ do not commute. Using these two matrices we introduce the matrices

$$\hat{V}_{jk} := \exp(i\theta_j\hat{n})\exp(-ik\hat{\theta}) \tag{12}$$

where $j, k = 0, 1, \ldots, N-1$ and $V_{00} = I_N$. Obviously these N^2 matrices are unitary. Inserting (9), (10), and (11) into (12) we can write

$$\hat{V}_{jk} := \sum_{m=0}^{N-1} \exp(i\theta_j m)|\phi_{m \bmod N}\rangle\langle|\phi_{m+k \bmod N}|. \tag{13}$$

Let

$$|\Phi\rangle := \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |\phi_k\rangle \otimes |\phi_k\rangle \tag{14}$$

where $\otimes$ denotes the Kronecker product of matrices (Steeb, 1997). This state is independent of the chosen basis in each subsystem. Using this state we define the N^2 states

$$|\Phi_{jk}\rangle := (\hat{V}_{jk} \otimes I_N)|\Phi\rangle \equiv (\exp(i\theta_j \hat{n}) \exp(-ik\hat{\theta}) \otimes I_N)\Phi\rangle \tag{15}$$

where $|\Phi\rangle = |\Phi_{00}\rangle$. It can easily be shown that the N^2 vectors $|\Phi_{jk}\rangle$ form an orthonormal basis in the Hilbert space $\mathbf{C}^{N^2}$, i.e. we have $\langle\Phi_{jk}|\Phi_{mn}\rangle = \delta_{jm}\delta_{kn}$ and

$$\sum_{j=0}^{N-1} \sum_{k=0}^{N-1} |\Phi_{jk}\rangle\langle\Phi_{jk}| = I_N \otimes I_N = I_{N^2}.$$

The measure for entanglement for pure states $E(|\psi\rangle\langle\psi|)$ is defined as follows (Hardy and Steeb, 2001; Nielsen and Chuang, 2000; Preskill, 2000; Steeb and Hardy, 2000, 2002)

$$E(|\psi\rangle\langle\psi|) := S_{\dim(\mathcal{H}_1)}(\rho_{\mathcal{H}_1}) = S_{\dim(\mathcal{H}_2)}(\rho_{\mathcal{H}_2}) \tag{16}$$

where $\mathcal{H}_1 = \mathcal{H}_2 = \mathbf{C}^N$ and the density operators are defined as

$$\rho_{\mathcal{H}_1} := \mathrm{Tr}_{\mathcal{H}_2}|\psi\rangle\langle\psi|, \quad \rho_{\mathcal{H}_2} := \mathrm{Tr}_{\mathcal{H}_1}|\psi\rangle\langle\psi| \tag{17}$$

and $S_b(\rho) := -\mathrm{Tr}\rho \log_b \rho$. Tr denotes the trace and $\mathrm{Tr}_{\mathcal{H}_1}$ denotes the partial trace over $\mathcal{H}_1$. We use the base b for the logarithm $\log_b$. We have $0 \log_b 0 = 0$ and $1 \log_b 1 = 0$. Thus $0 \le E \le 1$. If $E = 1$ we call the pure state maximally entangled. If $E = 0$, the pure state is not entangled. We note that

$$S_b(\rho) := -\sum_{j=1}^{k} \lambda_j \log_b \lambda_j$$

where $\{\lambda_j : j = 1, \ldots, k\}$ are the eigenvalues of ρ and ρ is a linear operator on a k-dimensional Hilbert space.

Using this definition for entanglement of pure states we find that the states $|\Phi_{jk}\rangle$ are maximally entangled. We obtain

$$|\Phi_{jk}\rangle\langle\Phi_{jk}| = \frac{1}{N} \sum_{l=0}^{N-1} \sum_{m=0}^{N-1} \exp(i\theta_j(l-m))|\phi_{l+k}\rangle\langle\phi_{m+k}| \otimes |\phi_l\rangle\langle\phi_m|. \tag{18}$$

Taking the partial trace over the first system yields

$$\rho_2 := \mathrm{Tr}_1(|\Phi_{jk}\rangle\langle\Phi_{jk}|) = \frac{1}{N}\sum_{l=0}^{N-1}|\phi_l\rangle\langle\phi_l|. \tag{19}$$

Thus $S_N(\rho_2) = 1$. The state $|\Phi\rangle$ given by (14) is a maximally entangled state. Thus the other states are obtained by a local unitary transformation (15) from the maximally entangled one.

For $N = 2$ we find the Bell states given above. For $N = 3$ we obtain the nine states

$$\frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$$

$$\frac{1}{\sqrt{3}}\left(|00\rangle + e^{i\frac{2\pi}{3}}|11\rangle - e^{i\frac{\pi}{3}}|22\rangle\right), \qquad \frac{1}{\sqrt{3}}\left(|00\rangle - e^{i\frac{\pi}{3}}|11\rangle + e^{i\frac{2\pi}{3}}|22\rangle\right)$$

$$\frac{1}{\sqrt{3}}(|10\rangle + |21\rangle + |02\rangle), \qquad \frac{1}{\sqrt{3}}(|01\rangle + |12\rangle + |20\rangle)$$

$$\frac{1}{\sqrt{3}}\left(|02\rangle + e^{i\frac{2\pi}{3}}|10\rangle - e^{i\frac{\pi}{3}}|21\rangle\right), \qquad \frac{1}{\sqrt{3}}\left(|02\rangle - e^{i\frac{\pi}{3}}|10\rangle + e^{i\frac{2\pi}{3}}|21\rangle\right)$$

$$\frac{1}{\sqrt{3}}\left(|01\rangle + e^{i\frac{2\pi}{3}}|12\rangle - e^{i\frac{\pi}{3}}|20\rangle\right), \qquad \frac{1}{\sqrt{3}}\left(|01\rangle - e^{i\frac{\pi}{3}}|12\rangle + e^{i\frac{2\pi}{3}}|20\rangle\right)$$

where $|00\rangle = |0\rangle \otimes |0\rangle$ etc and $|0\rangle, |1\rangle, |2\rangle$ denote the standard basis in $\mathbf{C}^3$.

Given an entangled state in $\mathbf{C}^N \otimes \mathbf{C}^N$ it is important to know if it can be distilled, i.e. r copies of it can be transformed by local operations and classical communication into s copies of $|\Phi\rangle$. State distillability, or useful quantum correlations, offer an alternative way of analyzing quantum nonlocality. All bipartite entangled pure states can be reversibly transformed using local operations and classical communication into $|\Phi\rangle$ (in the so-called asymptotic regime).

The set of the N^2 projection matrices

$$\{X_{jk} = |\Phi_{jk}\rangle\langle\Phi_{jk}| : j, k = 0, 1, \ldots, N-1\}$$

describe the generalized Bell measurement. An application of the Bell measurement is in teleportation. For any matrix

$$\hat{O} = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} O_{mn}|\phi_m\rangle\langle\phi_n|$$

we have

$$\sum_{j=0}^{N-1}\sum_{k=0}^{N-1}\hat{V}_{jk}\hat{O}\hat{V}_{jk}^{*} = N(\mathrm{Tr}\hat{O})I_N. \tag{20}$$

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